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INTERNAL FE APPROXIMATION OF SPACES OF DIVERGENCE-FREE FUNCTIONS IN THREE-DIMENSIONAL DOMAINS

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SUMMARY

The space of divergence-free vector functions with vanishing normal flux on the boundary is approximated by subspaces of finite elements having the same property. An easy way of generating basis functions in these subspaces is shown.

KEY WORDS Divergence-free Functions Finite Elements Internal Approximation Stream Function

INTRODUCTION

The purpose of this paper is to construct finite element subspaces of the spaces of divergence-free functions. This is a frequently appearing and important step in a numerical simulation of some phenomena in continuum mechanics, electromagnetism, heat and fluid flow problems, etc. Finite element spaces of functions, the divergences of which exist in the sense of distributions, and various degrees of freedom (parameters) of these spaces are given, for instance, in References 1–4. However, adding the equilibrium condition $\operatorname{div} \mathbf{q} = 0$, we obtain constraints among the parameters of each element.[5] These constraints can be removed, for instance by the method of Lagrange multipliers,[6,7] but this partly complicates a computational process. Therefore, the condition $\operatorname{div} \mathbf{q} = 0$ is mostly satisfied only approximated,[1,8–12] for instance by least squares methods, penalty methods, or the integral of the divergence over each element is required to be zero, etc. However, any 'external' approximation of $\operatorname{div} \mathbf{q} = 0$ does not allow one to establish any upper (or lower) bound of the critical value of an energy functional. On the other hand, conforming FE methods,[6] which are based on internal approximations, do not have this disadvantage. Moreover, the simultaneous use of the principles of minimum potential and complementary energy, e.g. to elliptic problems, yields even two-sided energy bounds[13] when using conforming FE methods. One can obtain also *a posteriori* error bounds and apply the hypercircle method. More details about these benefits of internal approximations of spaces of divergence-free functions can be found in References 13 and 14.

In this paper we shall describe an internal finite element approximation of the following space

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which appears in variational formulations of some considerable problems:[9,12,15–17]

$$H_0(\mathrm{div}^0;\Omega)=\{\mathbf{q}\in(L^2(\Omega))^d \mid (\mathbf{q},\mathrm{grad}\,z)_0=0,\quad \forall z\in H^1(\Omega)\}. \tag{1}$$

Here $\Omega\subset\mathbb{R}^d$, $d=2,3$, is a bounded domain with a Lipschitz boundary,[13] $(.,.)_0$ is the inner product in $(L^2(\Omega))^l$, $l=1,2,3$, and $H^k(\Omega)$ is the usual Sobolev space with the norm $\|\cdot\|_k$. Note that for smooth $\mathbf{q}\in H_0(\mathrm{div}^0;\Omega)$, by the Green formula (see (7) below)

$$\mathrm{div}\,\mathbf{q}=0 \quad \text{in} \quad \Omega, \qquad \mathbf{n}\cdot\mathbf{q}=0 \quad \text{on} \quad \partial\Omega, \tag{2}$$

where $\mathbf{n}$ is the exterior unit normal to the boundary $\partial\Omega$.

Let us recall (Reference 9, p. 25), that for two-dimensional domains

$$H_0(\mathrm{div}^0;\Omega)=\mathrm{curl}\{w\in H^1(\Omega)\mid \mathbf{n}\cdot\mathrm{curl}\,w=0 \text{ on } \partial\Omega\}, \tag{3}$$

where $\mathrm{curl}\,w=(\partial_2 w,-\partial_1 w)$.

Assume for a moment that $\Omega\subset\mathbb{R}^2$ is simply connected and let $W_h\subset H_0^1(\Omega)$ be an arbitrary finite element space. Here $H_0^1(\Omega)$ is the closure of $C_0^\infty(\Omega)$ in the $\|\cdot\|_1$ norm. Defining the space (see Reference 18, p. 44)

$$Q_h=\mathrm{curl}\,W_h, \tag{4}$$

we see by (3) that $Q_h\subset H_0(\mathrm{div}^0;\Omega)$, and thus Q_h is called the space of equilibrium finite elements.

Obviously, if $\{w^i\}$ is a basis in W_h then $\{\mathrm{curl}\,w^i\}$ is a basis in Q_h, since the linear mapping

$$\mathrm{curl}\colon W_h\to Q_h \tag{5}$$

is bijective (its kernel is clearly zero). Moreover, for supports we obtain $\mathrm{supp}\,w^i\supseteq\mathrm{supp}\,\mathrm{curl}\,w^i$, i.e. the number of arithmetic operations is really very small when computing, for instance, scalar products of divergence-free basis functions. This method has recently been analysed and numerically tested[16,18] and in this paper we give some generalizations to the three-dimensional case. Another approach to the description of an internal FE approximation of the space $H_0(\mathrm{div}^0;\Omega)$ for $\Omega\subset\mathbb{R}^3$ is presented in Reference 17. The exact fulfilment of the condition $\mathrm{div}\,\mathbf{q}=0$ in the space of finite elements for two-dimensional problems can be found for instance in References 1, 7, 12, 19 and 20.

SPACES OF EQUILIBRIUM FINITE ELEMENTS IN $\mathbb{R}^3$

First we introduce a space[3,4,9] of vector functions the divergence of which exists in the sense of distributions

$$H(\mathrm{div};\Omega)=\{\mathbf{q}\in(L^2(\Omega))^3\mid \exists f\in L^2(\Omega)\colon(\mathbf{q},\mathrm{grad}\,z)_0+(f,z)=0,\forall z\in H_0^1(\Omega)\},$$

and its subspace of divergence-free (solenoidal) functions

$$H(\mathrm{div}^0;\Omega)=\{\mathbf{q}\in(L^2(\Omega))^3\mid(\mathbf{q},\mathrm{grad}\,z)_0=0,\forall z\in H_0^1(\Omega)\}.$$

Since the test functions z vanish on the boundary $\partial\Omega$, there are no conditions upon the normal flux $\mathbf{n}\cdot\mathbf{q}$ on $\partial\Omega$ (cf. (1) and (2)). Let now $\mathbf{w}=(w_1,w_2,w_3)\in(H^1(\Omega))^3$ and $z\in C_0^\infty(\Omega)$ be arbitrary. Then by the Green theorem

$$(\mathrm{curl}\,\mathbf{w},\mathrm{grad}\,z)_0=(\mathbf{w},\mathrm{curl}\,\mathrm{grad}\,z)_0=0,$$

where $\mathrm{curl}\,\mathbf{w}=(\partial_2 w_3-\partial_3 w_2,\partial_3 w_1-\partial_1 w_3,\partial_1 w_2-\partial_2 w_1)$. Hence, using the density $H_0^1(\Omega)=\overline{C_0^\infty(\Omega)}$, we obtain

$$\mathrm{curl}\,\mathbf{w}\in H(\mathrm{div}^0;\Omega), \quad \text{for } \mathbf{w}\in(H^1(\Omega))^3. \tag{6}$$

We further recall (Reference 9, p. 16), that the functional $\mathbf{q} \rightarrow \mathbf{n} \cdot \mathbf{q}|_{\partial\Omega}$ defined on $(C^\infty(\bar{\Omega}))^3$ can be extended by continuity to a linear continuous mapping from the space $H(\text{div};\Omega)$ into $H^{-1/2}(\partial\Omega)$, the latter being the dual space to the space of traces $H^{1/2}(\partial\Omega)$ of functions from $H^1(\Omega)$. In this case, the Green formula is of the form

$$(\mathbf{q}, \text{grad}\, z)_0 + (\text{div}\, \mathbf{q}, z)_0 = \langle \mathbf{n} \cdot \mathbf{q}, z \rangle_{\partial\Omega}, \forall \mathbf{q} \in H(\text{div};\Omega), \forall z \in H^1(\Omega), \tag{7}$$

where $\langle \cdot, \cdot \rangle_{\partial\Omega}$ denotes the duality pairing between $H^{-1/2}(\partial\Omega)$ and $H^{1/2}(\partial\Omega)$. Introducing the space

$$W = \{\mathbf{w} = (w_1, w_2, w_3) \in (H^1(\Omega))^3 \,|\, w_3 = 0 \quad \text{in}\, \Omega, \mathbf{n} \cdot \text{curl}\, \mathbf{w} = 0 \quad \text{on}\, \partial\Omega\}, \tag{8}$$

we prove the following theorem (cf. (3)).

Theorem 1

Let $\Omega \subset \mathbb{R}^3$ be a bounded domain with a Lipschitz boundary. Then

$$H_0(\text{div}^0;\Omega) = \text{curl}\, W.$$

Proof. Let $\mathbf{q} \in H_0(\text{div}^0;\Omega)$ be arbitrary. According to Reference 9, p. 29, there exists the so-called stream function $\mathbf{w} = (w_1, w_2, 0) \in (H^1(\Omega))^3$ (not uniquely determined) such that

$$\mathbf{q} = \text{curl}\, \mathbf{w}$$

and we immediately see that $\mathbf{n} \cdot \text{curl}\, \mathbf{w} = \mathbf{n} \cdot \mathbf{q} = 0$ on $\partial\Omega$.

Conversely, let $\mathbf{w} \in W$ be given. Then from the Green formula (7), (6) and (8) we obtain

$$(\text{curl}\, \mathbf{w}, \text{grad}\, z)_0 = (-\text{div}\, \text{curl}\, \mathbf{w}, z)_0 + \langle \mathbf{n} \cdot \text{curl}\, \mathbf{w}, z \rangle_{\partial\Omega} = 0, \quad \forall z \in H^1(\Omega),$$

that is $\text{curl}\, \mathbf{w} \in H_0(\text{div}^0;\Omega)$ by (1).

Remark

By References 9 and 21 there exists a stream function $\mathbf{w}'$ for $\mathbf{q} \in H_0(\text{div}^0;\Omega)$ so that $\mathbf{w}'$ is also divergence-free. However, this physically natural choice is not suitable for our purposes, as we shall see below, and thus in (8) we take simply $w_3 = 0$ (instead of $\text{div}\, \mathbf{w}' = 0$). The condition $w_3 = 0$ may be replaced, of course, by $w_1 = 0$ or $w_2 = 0$.

Now, let $W_h \subset W$ be an arbitrary finite element space whose functions are continuous and piecewise polynomial on some partition of $\bar{\Omega}$ (h is the usual mesh parameter). Analogously to (4) we define the space of equilibrium finite elements as

$$Q_h = \text{curl}\, W_h \tag{9}$$

and from Theorem 1 we see again that $Q_h \subset H_0(\text{div}^0;\Omega)$.

Corollary

Let $\{W_h\}$ be a system of finite element subspaces of W such that the union $\cup_h W_h$ is dense in W with respect to the $\|\cdot\|_1$ norm. Then $\cup_h Q_h$ is dense in $H_0(\text{div}^0;\Omega)$ in the $\|\cdot\|_0$ norm.

Proof. For a given $\mathbf{q} \in H_0(\text{div}^0;\Omega)$ there exists by Theorem 1 a function $\mathbf{w} \in W$ (not uniquely determined) such that $\mathbf{q} = \text{curl}\, \mathbf{w}$. Let $\mathbf{w}_h \in W_h$ be such that $\mathbf{w}_h \rightarrow \mathbf{w}$ in the $\|\cdot\|_1$ norm and let us define $\mathbf{q}_h = \text{curl}\, \mathbf{w}_h \in Q_h$. Then

$$\|\mathbf{q} - \mathbf{q}_h\|_0 = \|\text{curl}(\mathbf{w} - \mathbf{w}_h)\|_0 \leqslant \|\mathbf{w} - \mathbf{w}_h\|_1 \rightarrow 0 \quad \text{for}\ h \rightarrow 0.$$

Note that Theorem 1 allows one to give a description of any subspace of $H_0(\mathrm{div}^0;\Omega)$. So let $Q' \subset H_0(\mathrm{div}^0;\Omega)$ be an arbitrary subspace (e.g. a finite element subspace). Then by Theorem 1, Q' must be always of the form $Q' = \mathrm{curl}\, W'$ for a suitable subspace $W' \subset W$.

BASIS FUNCTIONS IN THE SPACE OF EQUILIBRIUM ELEMENTS

In general, the mapping curl: $W_h \to Q_h$ is unfortunately not bijective in $\mathbb{R}^3$ (as it was in (5)). This would bring some difficulties with defining basis functions in Q_h. Therefore, we shall now construct a subspace $V_h \subset W_h$ such that

$$\mathrm{curl}: V_h \to Q_h \tag{10}$$

is bijective.

For simplicity, we shall give first some restrictions.

Definition

A domain $\Omega \subset \mathbb{R}^3$ is said to belong to the class $\mathscr{L}$ if

(i) it is a bounded domain with a Lipschitz boundary,
(ii) there exists a simply connected domain $\omega \subset \mathbb{R}^2$ and a positive function $F: \omega \to \mathbb{R}^1$ (in general discontinuous) such that

$$\Omega = \{(x_1, x_2, x_3) \in \mathbb{R}^3 \mid (x_1, x_2) \in \omega, 0 < x_3 < F(x_1, x_2)\}.$$

In order to introduce V_h we further define

$$\partial\Omega_0 = \omega \times \{0\} \subset \partial\Omega.$$

We see that $\partial\Omega_0$ lies in the $(x_1, x_2, 0)$ plane and it is the 'lowest' part of the boundary $\partial\Omega$. For instance, a cube, cone or cylinder (with the base $\partial\Omega_0$) are in $\mathscr{L}$.

Henceforth, we shall require the following property of the space W_h:

$$\mathbf{w} \in W_h \Rightarrow \mathring{\mathbf{w}} \in W_h, \tag{11}$$

where $\mathring{\mathbf{w}}$ is defined as

$$\mathring{\mathbf{w}}(x_1, x_2, x_3) = \mathbf{w}(x_1, x_2, 0), \quad (x_1, x_2, x_3) \in \Omega, \tag{12}$$

(the right-hand side is the trace of $\mathbf{w}$ on $\partial\Omega_0$). Later we show that $\mathrm{curl}\,\mathring{\mathbf{w}} = 0$ in Ω (cf. (14) below). Hence, $\mathbf{n}\cdot\mathrm{curl}\,\mathring{\mathbf{w}} = 0$ on $\partial\Omega$ and the condition (11) can be easily satisfied when employing e.g. prismatic or rectangular C^0-elements.[6,22]

Theorem 2

Let $\Omega \in \mathscr{L}$ and let $W_h \subset W$ satisfy (11). Then for

$$V_h = \{\mathbf{v} \in W_h \mid \mathbf{v} = 0 \text{ on } \partial\Omega_0\},$$

the mapping (10) is bijective.

Proof.

Injectivity

Assume that $\mathrm{curl}\,\mathbf{v} = 0$ for some $\mathbf{v} \in V_h$. As $\Omega \in \mathscr{L}$, it is obvious that Ω is simply connected.

Consequently, there exists a scalar function $s \in H^1(\Omega)$ (unique apart from an additive constant) such that[21,23]

$$\mathbf{v} = \operatorname{grad} s.$$

However, $\mathbf{v} \in V \subset (H^1(\Omega))^3$ (i.e. $\partial_i s \in H^1(\Omega)$, $i = 1,2,3$) and thus we even get $s \in H^2(\Omega)$. By the Sobolev embedding theorem, s is continuous on $\bar{\Omega}$ (see, for instance, Reference 9, p. 111). It is

$$\begin{aligned} s(x_1, x_2, x_3) &= s(x_1, x_2, 0) + \int_0^{x_3} \partial_3 s(x_1, x_2, \xi) \mathrm{d}\xi \\ &= s(x_1, x_2, 0), \qquad x \in \Omega, \end{aligned} \tag{13}$$

since $\partial_3 s = v_3 = 0$. The condition $\mathbf{v} = 0$ on $\partial\Omega_0$ implies that s is constant on $\partial\Omega_0$. Hence, by (13) the function s is constant on the whole Ω and thus $\mathbf{v} = 0$.

Surjectivity

Let $\mathbf{q} \in Q_h$ be arbitrary. According to (9) there exists a continuous piecewise polynomial function $\mathbf{w} = (w_1, w_2, 0) \in W_h$ (not uniquely determined) such that

$$\mathbf{q} = \operatorname{curl} \mathbf{w}.$$

We set

$$\mathbf{v} = \mathbf{w} - \mathring{\mathbf{w}},$$

where $\mathring{\mathbf{w}}$ is defined in (12). Then $\mathbf{v} = 0$ on $\partial\Omega_0$ and from (11) we obtain that $\mathbf{v} \in V_h \subset W_h$. Moreover, we verify whether

$$\mathbf{q} = \operatorname{curl} \mathbf{v}$$

holds.

Clearly it suffices to prove that for $\mathring{\mathbf{w}} = (\mathring{w}_1, \mathring{w}_2, \mathring{w}_3)$ we have

$$\operatorname{curl} \mathring{\mathbf{w}} = 0 \quad \text{in } \Omega. \tag{14}$$

The first two components of the vector $\operatorname{curl} \mathring{\mathbf{w}}$ are zeros, because $\mathring{w}_3 = 0$ and by (12) $\partial_3 \mathring{w}_1 = \partial_3 \mathring{w}_2 = 0$ in Ω. Next, we show that also the third component vanishes. Let $\mathbf{x} = (x_1, x_2, x_3) \in \Omega$ be given and let us denote by $\mathbf{n}_0 = (0, 0, -1)$ the exterior unit normal to $\partial\Omega_0 \subset \partial\Omega$. Using now (12) and the fact that $\mathbf{n} \cdot \operatorname{curl} \mathbf{w} = 0$ on $\partial\Omega$ for $\mathbf{w} \in W_h$, we come to

$$\begin{aligned} \partial_1 \mathring{w}_2(\mathbf{x}) - \partial_2 \mathring{w}_1(\mathbf{x}) &= \partial_1 w_2(x_1, x_2, 0) - \partial_2 w_1(x_1, x_2, 0) \\ &= -\mathbf{n}_0 \cdot \operatorname{curl} \mathbf{w}(x_1, x_2, 0) = 0. \end{aligned}$$

Under the assumptions of Theorem 2, we can introduce basis functions of the space $Q_h = \operatorname{curl} V_h$.

Let $\{\mathbf{v}^i\}_{i=1}^m$ be a basis in V_h. Then evidently

$$\mathbf{q}^i = \operatorname{curl} \mathbf{v}^i, \qquad i = 1, \ldots, m, \tag{15}$$

are basis functions in $Q_h \subset H_0(\operatorname{div}^0; \Omega)$ since the linear mapping (10) is bijective (i.e. $\dim Q_h = \dim V_h$). Moreover, we see that

$$\operatorname{supp} \mathbf{q}^i \subseteqq \operatorname{supp} \mathbf{v}^i, \qquad i = 1, \ldots, m.$$

Of course, $\mathbf{q}^i$ are not in general continuous in the whole of $\bar{\Omega}$. However, one can easily prove by (7) that the normal component $\mathbf{n}_f \cdot \mathbf{q}^i$ is continuous at each common face f of any two adjacent elements, where $\mathbf{n}_f$ is a normal to f (see also Reference 2, p. 62).

Example

We introduce a typical shape of divergence-free basis functions derived from the usual trilinear elements, the ansatz-polynomials, which are of the form

$$c_0 + c_1x_1 + c_2x_2 + c_3x_3 + c_4x_1x_2 + c_5x_1x_3 + c_6x_2x_3 + c_7x_1x_2x_3$$

on every rectangular element.[6] Assume for simplicity that a uniform mesh (with the mesh size h) is given and let e.g. $\mathbf{y} = (0, h, h)$ be a nodal point in $\bar{\Omega}$. If $\mathbf{y} \notin \partial\Omega$ then we can have two standard basis functions $\mathbf{v}^i, \mathbf{v}^{i+1} \in V_h$ for some $i \in \{1, \dots, m\}$ such that

$$\operatorname{supp} \mathbf{v}^i = \operatorname{supp} \mathbf{v}^{i+1} = [-h, h] \times [0, 2h] \times [0, 2h].$$

This support consists of eight elements. One of them is, for instance, $K = [0, h] \times [0, h] \times [0, h]$, and we may immediately obtain

$$\begin{aligned} \mathbf{v}^i &= ((h - x_1)x_2x_3, 0, 0)/h^3 \qquad \text{in } K, \\ \mathbf{v}^{i+1} &= (0, (h - x_1)x_2x_3, 0)/h^3 \qquad \text{in } K. \end{aligned} \tag{16}$$

Now by (15) a direct calculation leads to

$$\begin{aligned} \mathbf{q}^i &= (0, (h - x_1)x_2, (x_1 - h)x_3)/h^3, \qquad \text{in } K, \\ \mathbf{q}^{i+1} &= ((x_1 - h)x_2, 0, -x_2x_3)/h^3, \qquad \text{in } K. \end{aligned} \tag{17}$$

Similarly we obtain $\mathbf{q}^i$ and $\mathbf{q}^{i+1}$ on the other seven elements of $\operatorname{supp} \mathbf{v}^i$.

Suppose further that $\mathbf{y} = (0, h, h) \in \partial\Omega$. For simplicity let $\Omega = (0, 1) \times (0, 1) \times (0, 1)$. In this case, (8) yields $\partial_2 v_3(\mathbf{y}) = \partial_3 v_2(\mathbf{y})$ for any $\mathbf{v} \in V_h$, since $(-1, 0, 0)$ is the exterior unit normal to $\partial\Omega$ at $\mathbf{y}$. Thus by (16) we find that $\mathbf{v}^i \in V_h (\mathbf{v}^{i+1} \notin V_h)$. The corresponding support of $\mathbf{q}^i$ will consist of four elements only and e.g. $\mathbf{q}^i|_K$ will be given by (17).

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